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Note

Ramsey Numbers for Quadrangles and Triangles

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The Ramsey numbers $r(mK_4, nK_3)$, where mK_4 consists of m disjoint complete quadrangles and nK_3 consists of n disjoint complete triangles, are calculated.

The graph mK_4 consists of m disjoint complete quadrangles and nK_3 consists of n disjoint triangles. The Ramsey number $r(mK_4, nK_3)$ is the smallest integer such that any two-colored graph with that number of vertices contains a subgraph mK_4 of the first color or a subgraph nK_3 of the second; its existence is guaranteed by Ramsey's theorem [2]. We will prove that these numbers are given by

$$\begin{aligned} r(mK_4, nK_3) &= 9 && \text{when } m = n = 1, \\ &= 3m + 3n + 1 && \text{when } n \geqslant m > 1, \\ &= 3n + 5 && \text{when } n \geqslant 2 \text{ and } m = 1, \\ &= 4m + 2n + 1 && \text{when } m \geqslant n \text{ and } m \geqslant 2. \end{aligned}$$

The proof is by induction on m and n based on the values $r(K_4, K_3) = 9$, $r(2K_4, K_3) = 11$, $r(K_4, 2K_3) = 11$, and $r(2K_4, 2K_3) = 13$. The first can be found in [4] and the others can be proved in a straightforward way involving consideration of the different ways in which the relevant complete graph can be colored. As the proof is long in the case of $r(2K_4, K_3)$ there seems little point in detailing them here; copies of them can be obtained from the authors on request.

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We will assume that the two colors we are interested in are red and green, with the quadrangles generally colored red and the triangles green. All graphs have their edges colored red and green.

1. Lower Bounds

In this section we will establish that the numbers given are lower bounds for the appropriate Ramsey numbers. In each case this involves describing a graph with one fewer vertex than the number given and having no red mK_4 or green nK_3.

(i) $r(mK_4, nK_3) \geqslant 3m + 3n + 1$ if $n \geqslant m > 1$.

Let G_1 be a complete graph with $3m - 1$ vertices and all edges colored red, G_2 a complete graph with $3n - 1$ vertices and all edges colored green, G_3 a graph with two vertices and a red edge, and suppose the three graphs are disjoint. Join each vertex of G_1 to each vertex of G_3 by a green edge and join each vertex of G_2 to each vertex of G_1 and G_3 by a red edge. The result is a complete graph with $3m + 3n$ vertices which has no subgraph mK_4 colored red and no subgraph nK_3 colored green.

(ii) $r(K_4, nK_3) \geqslant 3n + 5$ if $n \geqslant 2$.

Let G_1 be a complete graph with five vertices which has red and green edges but no triangle of either color, G_2 a complete graph with $3n - 1$ vertices and all edges colored green, and suppose the two graphs are disjoint. Join each vertex of G_1 to each vertex of G_2 by a red edge. The resulting complete graph has $3n + 4$ vertices but no subgraph K_4 colored red and no subgraph nK_3 colored green.

(iii) $r(mK_4, nK_3) \geqslant 4m + 2n + 1$ if $m \geqslant n$ and $m \geqslant 2$.

Let G_1 be a complete graph with $4m - 1$ vertices and all edges colored red, G_2 a complete graph with $2n - 1$ vertices and all edges colored green, G_3 a graph with two vertices and a red edge, and suppose the three graphs are disjoint. Join each vertex of G_1 to each vertex in G_2 and G_3 by a green edge and join each vertex of G_2 to each vertex of G_3 by a red edge. The result is a complete graph with $4m + 2n$ vertices which has no subgraph mK_4 colored red and no subgraph nK_3 colored green.

2. Three Lemmas

The lemmas proved here will be used in the induction in the next section. The first of them will enable us to subtract from certain graphs six vertices having among their joining edges both a red K_4 and a green K_3. These two intersecting subgraphs play the same role as the tree in [3] and the bowtie in [1].

LEMMA 1. *If a complete graph with edges colored red and green has as subgraphs a red quadrangle and a green triangle then it has as subgraphs a red quadrangle and a green triangle having a vertex in common.*

Proof. Assume the contrary. Then no vertex of the green triangle is joined to three vertices of the red quadrangle by three red edges, and so each is joined by at least two green edges to the red quadrangle. Hence there are at least six green edges joining the quadrangle and the triangle. Because of this at least one vertex of the quadrangle has two green edges joining it to the green triangle. These are then two edges of a green triangle having a vertex in common with the red quadrangle.

A result that seems to be well known is that, if $m \geqslant 2$, then

$$r(mG, H) \leqslant r((m-1)\, G, H) + \rho(G),$$

where $\rho(G)$ is the degree of the graph G. In the present context this has the consequences:

LEMMA 2. *If $m \geqslant 2$ then*

$$r(mK_4, nK_3) \leqslant r((m-1)\, K_4, nK_3) + 4.$$

LEMMA 3. *If $n \geqslant 2$ then*

$$r(mK_4, nK_3) \leqslant r(mK_4, (n-1)\, K_3) + 3.$$

3. UPPER BOUNDS

Finally we show that the numbers given are upper bounds for the Ramsey numbers. The proof is by induction based on the initial values given earlier and with the induction steps based on the three lemmas in the last section.

First, as $r(2K_4, K_3) = 11$ it follows from Lemma 2 by induction that $r(mK_4, K_3) \leqslant 4m + 3$ if $m \geqslant 2$.

We show next that if $m \geqslant n \geqslant 1$ and $m \geqslant 2$ then $r(mK_4, nK_3) \leqslant 4m + 2n + 1$. As starting values for an inductive proof we have $r(2K_4, 2K_3) = 13$ and $r(mK_4, K_3) \leqslant 4m + 3$. Our induction will be on n. Suppose that $n > 1$ and if $n = 2$ then $m > 2$ and let G be a graph with $4m + 2n + 1$ vertices. First suppose that G has no red quadrangle; by induction it then has a subgraph $(n-1)K_3$ colored green. Subtract the $3(n-1)$ vertices of this subgraph from G to get a graph G_0 with $(4m+2n+1)-3(n-1) = 4m-n+4$ vertices. As $m \geqslant n > 1$ this number is greater than 9 and G_0 has a red quadrangle or a green triangle. The former is excluded and the latter can be adjoined to the $(n-1)K_3$ to give a green nK_3, completing the induction step. Alternatively G does have a red quadrangle. As G has more than $4m+2(n-1)+1$ vertices it has an mK_4 colored red or an $(n-1)K_3$ colored green. In the first case the induction is complete and in the second it can be deduced from Lemma 1 that G has a red quadrangle and a green traingle with a vertex in common. The touching red quadrangle and green triangle make up a subgraph, G_1 say, which has six vertices. Removing this from G we get a subgraph, G_2 say, which has $4(m-1)+2(n-1)+1$ vertices. From our induction assumption G_2 has either an $(m-1)K_4$ colored red or an $(n-1)K_3$ colored green. Joining whichever it is to the K_4 or the K_3 of the same color in G_2 we get, in G, either an mK_4 colored red or an nK_3 colored green, as required.

Next consider the case $n \geqslant m > 1$. The result $r(mK_4, nK_3) \leqslant 3m+3n+1$ follows by induction on n from Lemma 3 using the initial values for $m = n$ established in the last paragraph.

Finally, if $m = 1$, Lemma 3 again and the initial value $r(K_4, 2K_3) = 11$ enable us to prove by induction that $r(K_4, nK_3) \leqslant 3n+5$ when $n \geqslant 2$.

This completes the proof.

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